Awake Complexity of Distributed Minimum Spanning Tree

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Outline

Problem Statement

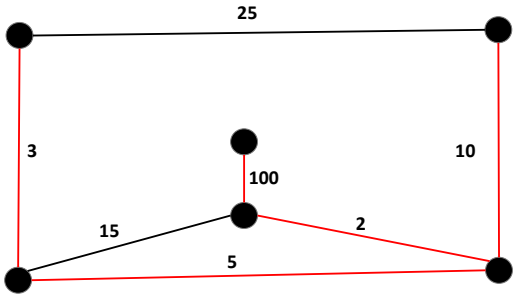
Final Goal

Find the **minimum spanning tree (MST)** of an input graph in the **CONGEST model** in an **awake efficient/energy efficient** manner.

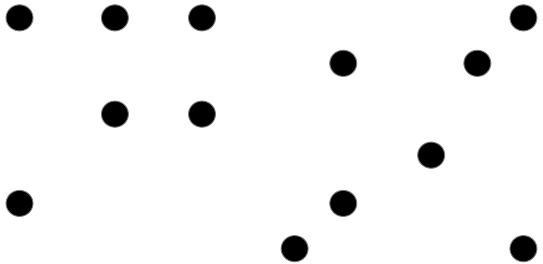
Minimum Spanning Tree (MST)

Minimum Spanning Tree

For a graph $G(V, E)$ where each edge $e \in E$ has associated weight, the minimum spanning tree of G is a connected subgraph of G that connects all nodes of V, has no cycles, and has the minimum sum of edge weights amongst all possible subgraphs satisfying the previous two conditions.



Distributed Computing Model



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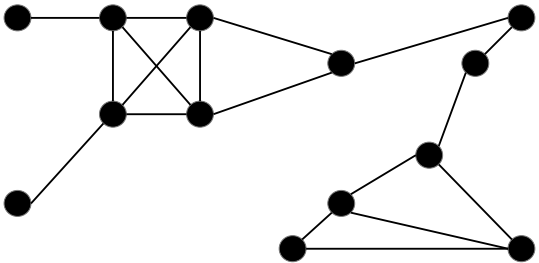

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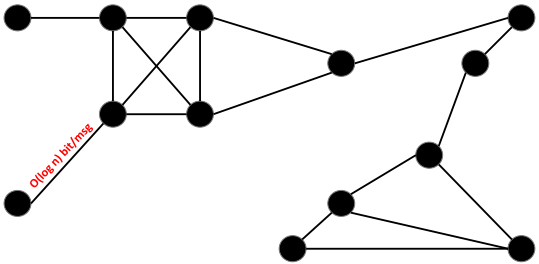
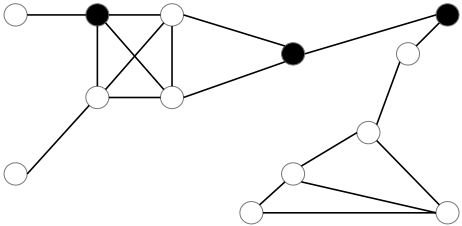


- n nodes with unique IDs, value of n known to all nodes
- Arbitrary graph
- CONGEST model: $O(\log n)$ bits per message over an edge

Awake/Energy Efficiency Defined & Measured



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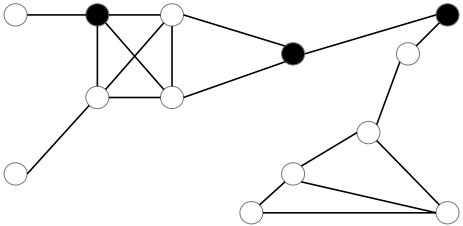

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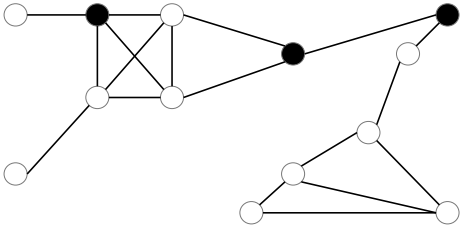

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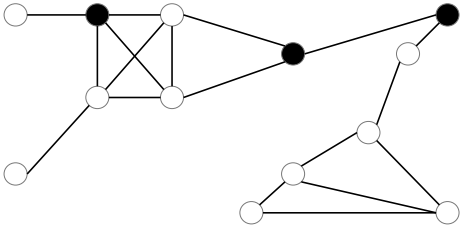

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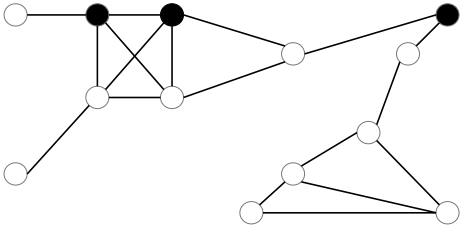

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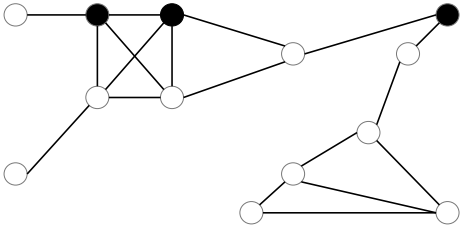

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- **Metric 1:** awake time; **Metric 2:** running time/round complexity

Outline

Why Care About Awake Efficient MST in CONGEST

1. MST long studied in CONGEST. Approx. 40 years now.
   Now established that $\Theta(D + \sqrt{n})$ is tight (up to log factors) bound for running time.
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6. Note that lots of work on energy complexity in RADIO model as well.

Our Contributions

Table: Summary of our Results.

Algorithm	Type	Awake Time (AT)	Run Time (RT)	AT Lower Bound	AT $\times$ RT Lower Bound
*Randomized-MST	Randomized	$O(\log n)$	$O(n \log n)$	$\Omega(\log n)$	$\tilde{\Omega}(n)$†
Deterministic-MST	Deterministic	$O(\log n)$	$O(n \log^5 n)$	$\Omega(\log n)$	$\tilde{\Omega}(n)$†
Modified Deterministic-MST	Deterministic	$O(\log n \log^* n)$	$O(n \log n \log^* n)$	$\Omega(\log n)$	$\tilde{\Omega}(n)$†
*♣Trade-Off-MST	Randomized	$\tilde{O}(n/2^k)$	$\tilde{O}(D + 2^k + n/2^k)$	-	-

*The algorithm outputs an MST with high probability.
♣This algorithm takes integer k as an input parameter.
†Holds for (some) graphs with diameter $\tilde{\Omega}(\sqrt{n})$.
Our lower bounds also apply to Monte Carlo randomized algorithms with constant success probability.
n is the number of nodes in the network and N is an upper bound on the largest ID of a node.

Why Care About This Talk

1. Technique for lower bound w.r.t. awake complexity.
2. Techniques for designing algs with low awake complexity.
3. In particular, one concept (LDT) used in to other work [Dufoulon et al., PODC 2023]. Potential applications to other problems.

Outline

Product (of Awake and Run Time Complexities) Lower Bound

Theorem Statement (Simplified)

For graphs with diameter $D = \omega(\sqrt{n}\log^3 n)$, any randomized alg. $\mathcal{A}$ that solves MST that is correct with probability $1 - \epsilon$ for any small fixed $\epsilon > 0$ has the following property:
If the running time of $\mathcal{A}$ is some $o(c)$, for some c between $\tilde{\Omega}(\sqrt{n})$ to $\tilde{O}(n)$, then awake time is at least $\tilde{\Omega}(n/c)$.

Example Implication

For any algorithm $\mathcal{A}$ that solves MST and runs in time $o(n^{3/4})$, its awake time must be at least $\tilde{\Omega}(n^{1/4})$.

Proof Sketch

Problems Considered

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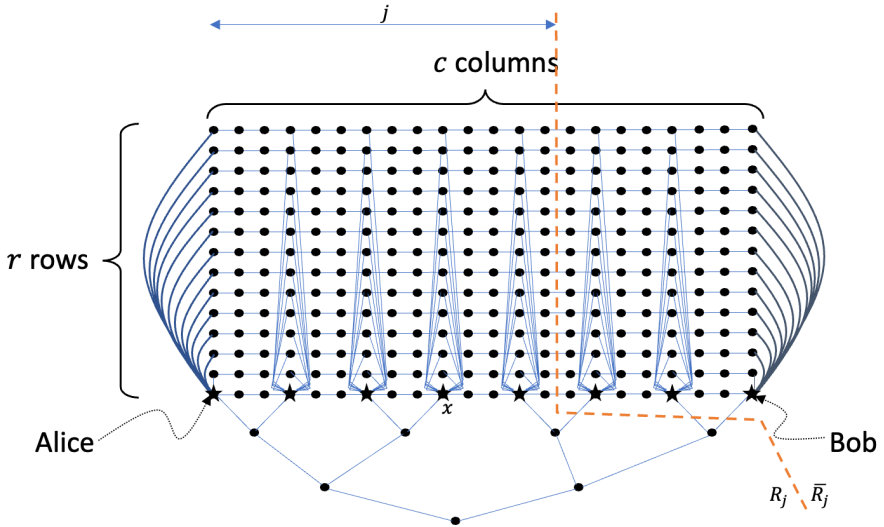
We focus on steps 1 & 2 in this talk.

Steps 1 & 2



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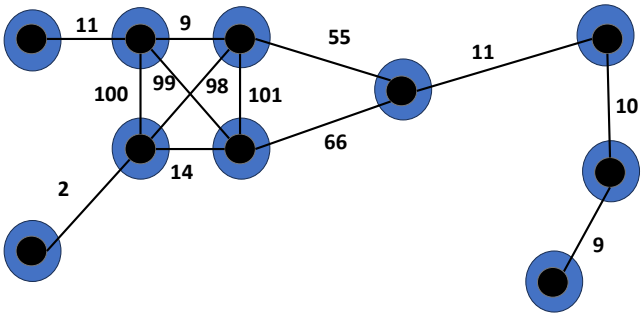
Intuition Behind Algorithms



- Algs are variants of classical GHS alg, modified to minimize awake complexity.
- Clusters (initially each node) merge using MOEs into larger clusters.
- Each phase, constant fraction of clusters merge. Totally $O(\log n)$ phases.

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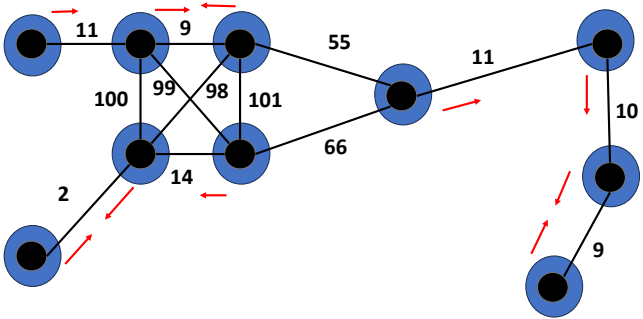

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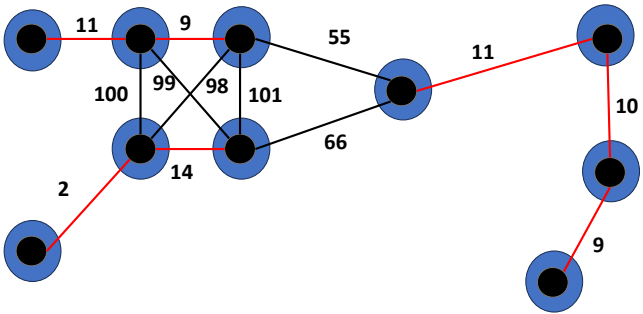

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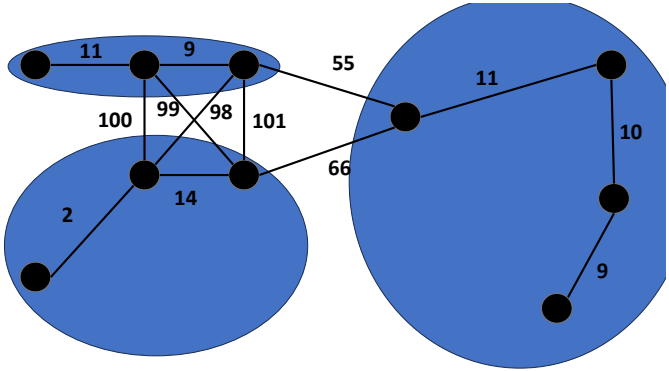

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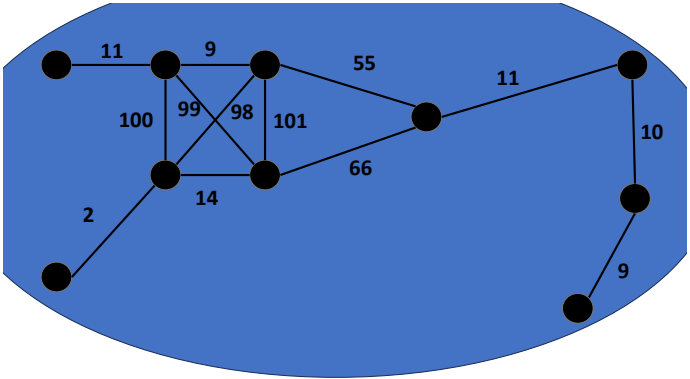

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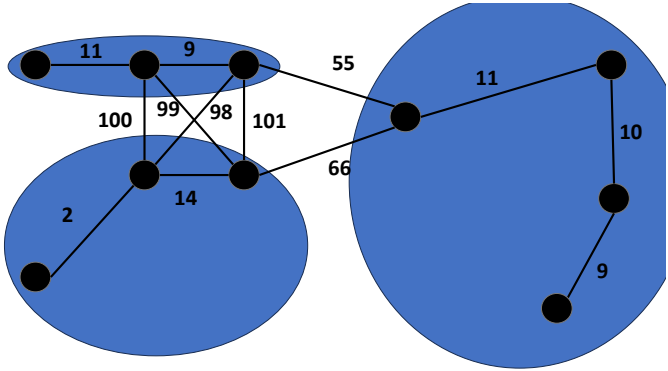

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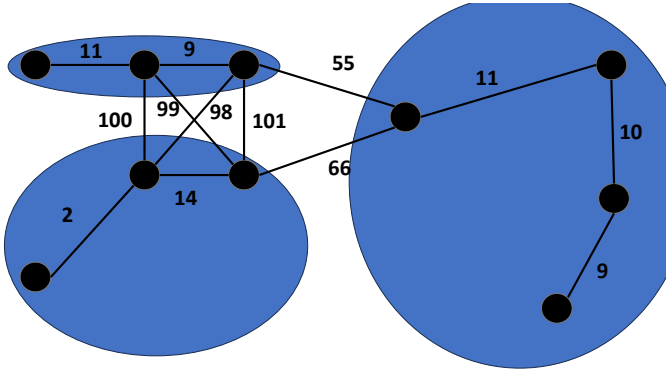

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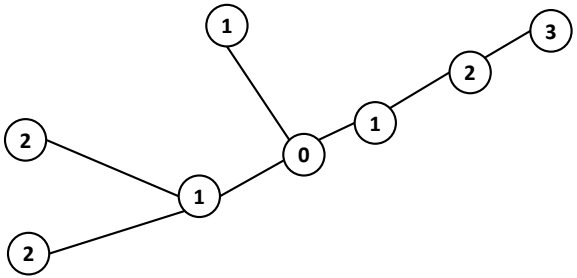

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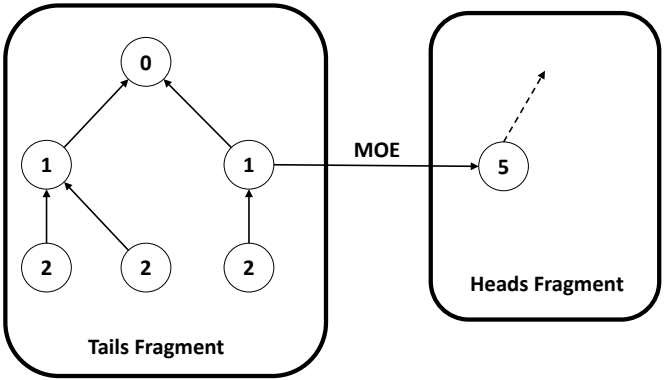
- **Idea:** Labeled Distance Tree (LDT) paired with transmission schedules of size $2n + 1$.

Randomized Algorithm



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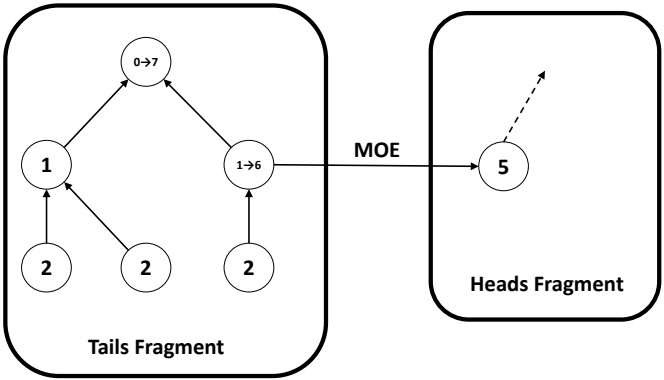

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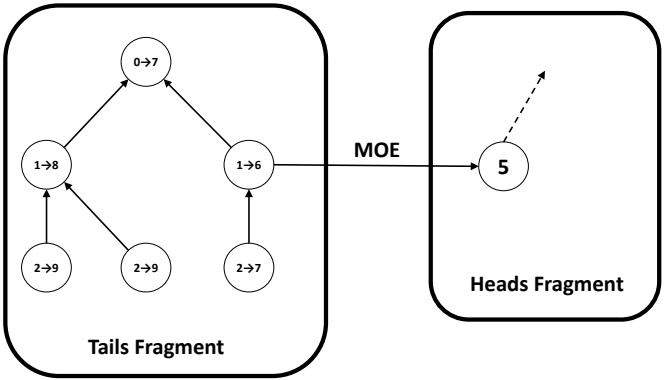

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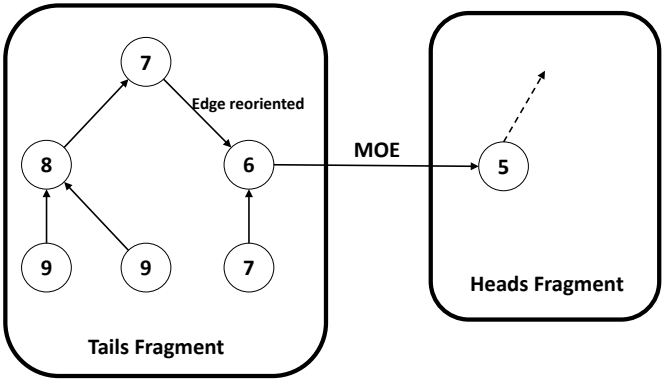

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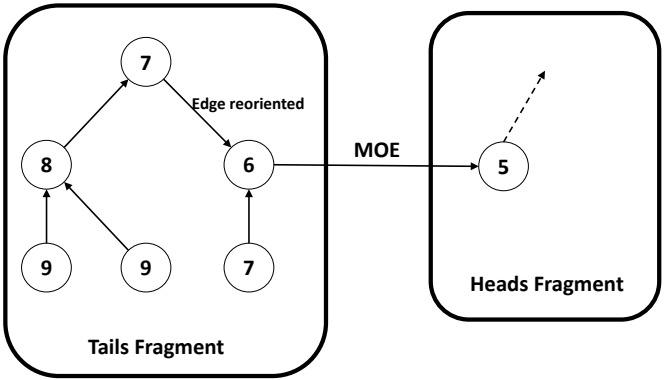

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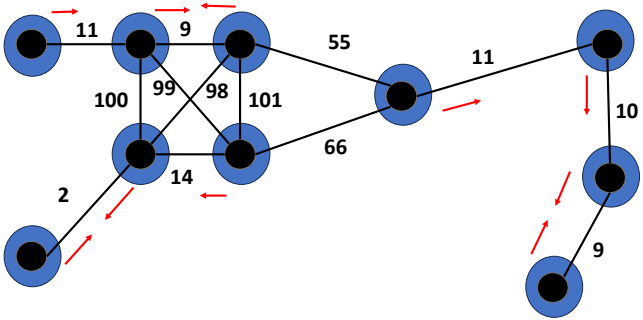

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- **Hint:** we want $O(1)$ awake time per node per phase.

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- Each phase, $O(1)$ awake complexity per node and $O(n)$ run time.
- **Result:** MST constructed w.h.p. in $O(\log n)$ awake complexity and $O(n \log n)$ round complexity.

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- **Solution:** Combinatorial argument:
  Take highest priority color.
  In any 5 coloring of subgraph, there are a constant fraction of nodes colored with it.
  Those nodes merge in that phase.

Deterministic Algorithm

- **Problem:** How to ensure no "long lines" without randomness?
- **Solution:** Use maximal matching (coloring with a small palette + matching).
- **Issue 1:** How to guarantee small palette enough?
- **Solution:** Restrict "valid" MOEs to be only at most 4 (3 incoming + 1 outgoing). Color palette 5 enough now.
- **Issue 2:** How to argue constant fraction of fragments merged in each phase?
- **Solution:** Combinatorial argument:
  Take highest priority color.
  In any 5 coloring of subgraph, there are a constant fraction of nodes colored with it.
  Those nodes merge in that phase.
- **Result 1:** MST in $O(\log n)$ awake time and $O(n \log^5 n)$ run time.
- **Result 2:** MST in $O(\log n \log^* n)$ awake time and $O(n \log n \log^* n)$ run time.

Outline

Conclusions & Future Work

Conclusions

1. In this talk, we looked at product lower bound and algorithms to find MST.
2. In paper, you can find the trade-off family of algorithms & unconditional awake time lower bound.

Future Work

1. Awake complexity of other problems.
2. Looking at awake complexity in presence of faults and/or dynamism of graph.

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